Imperial College
London

Lecture 9

Combining Signals & Systems together using IMU as an example

Peter Cheung
Dyson School of Design Engineering

URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

In this lecture, I will highlight SEVEN most important lessons about systems – the topic in the previous week. One objective is to provide you with a "birds eye's" view of this topic before I start the new topic next week on discrete systems.

I will also integrate everything I have covered so far on signals and systems using the example of the IMU module. IMU stands for Inertia Measurement Unit. I will consider the accelerometer and gyroscope data produced by the IMU, and show you how you can estimate the pitch and roll angles from these readings using a system (known as Complementary Filter).

7 things you have learned about systems (1)

1. A system can be characterised in terms of **differential equations** in the time-domain, or in terms of **transfer functions** in the Laplace domain.

2. **Laplace transform** converts differential equations into **algebraic equations**. While in Fourier, we use the frequency variable $j\omega$, in Laplace, we use the complex frequency variable s, where $s = \alpha + j\omega$.

3. **Transfer function** H(s) is the relationship between output Y(s) and input X(s) in the Laplace domain:

$$Y(s) = H(s) \times X(s)$$

4. **Frequency response** $H(j\omega)$ = Transfer function H(s) evaluated at $s = j\omega$:

$$H(j\omega) = H(s)\Big|_{s=j\omega}$$

In this course, I started with signals in the first five lectures. Then I moved to the analysis of systems (and these systems are designed to process the signals). Here are the first four of the most important lessons that you are meant to have grasped.

Remember, it is not so important to memorise anything, particularly don't memorise formulae – you can always look these up. However, the conceptual ideas behind each of these points are important.

7 things you have learned about systems (2)

5. Fourier transform and frequency response are only valid of **steady-state conditions**; Laplace transform and transfer function are useful for **both steady state and transient conditions**.
6. **Non-linear systems** can be approximated as linear if you operate over a small signal range.
7. A general **2^{nd} order system** can be expressed in term of **damping factor** ζ and **resonant frequency** ω_0, and it can be under-damped, critically- damped or over-damped.

Topics omitted:
- Ideas of poles and zeroes of a system
- How poles and zeroes affect steady-state and transient behaviour of a system
- Stability issues of a system

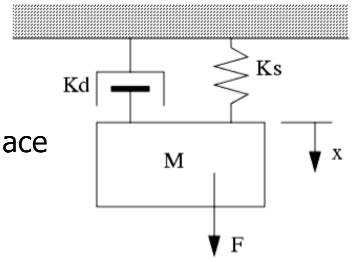
Here are the remaining points what have been covered so far. I will now go through each of these SEVEN points in the following slides.

Furthermore, there are a few issues that I have deliberately omitted this year. I may or may not cover them in later lectures, depending on how much time I have got left.

1 & 2 - Differential Equation vs Transfer Function

- We use differential equations to model systems in time-domain:

$$M \ddot{x}(t) + K_d \dot{x}(t) + K_s x(t) = F(t)$$



- We use transfer functions to model systems in Laplace domain as algebraic equation:

$$Ms^2 X(s) + K_d sX(s) + K_s X(s) = F(s)$$

$$(Ms^2 + K_d s + K_s)X(s) = F(s)$$

$$\Rightarrow H(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$\boxed{s^k \longleftrightarrow \frac{d^k}{dt^k}}$$

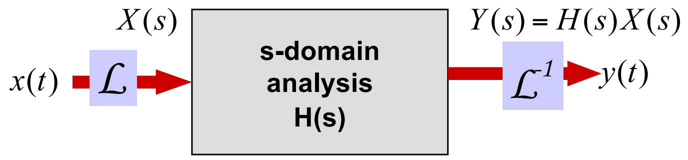
The first two lessons are related.

In time domain, we are familiar with using differential equations to model a physical system. Above is the differential equation modelling a mass connected to a fixed ceiling with a spring and a damper. The relative between the force applied F (input) and the vertical displacement x (output) is given in the 2nd order differential equation.

If we apply Laplace transform to the input, output and the differential equation, mapping d/dt to s, and d^2/dt^2 to s^2, we obtain the transfer function of the system in the Laplace variable s as shown.

H(s) is known as the transfer function of the system.

3 - Transfer Function H(s) = Y(s) / X(s)



- Once transformed to the s-domain, analysis and prediction of the system become easy if we know the system's characteristic H(s), which is also called the **transfer function**.

- Transfer function H(s) = Output Y(s) / Input X(s)

or

Output Y(s) = Transfer Function H(s) x Input X(s)

The transfer function is very useful concept. You transfer the input X(s) to the output Y(s) through this function H(s).

Therefore, through very simple multiplication, we can predict the output given the input:

Y(s) = H(s) X(s).

4 – Frequency Response vs Transfer Function

Laplace Transform

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

$s = j\omega$

Fourier Transform

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Transfer Function

$$H(s)$$

$s = j\omega$

Frequency Response

$$H(j\omega)$$

- We can find the frequency response of a system by substituting s = jω into the transfer function:

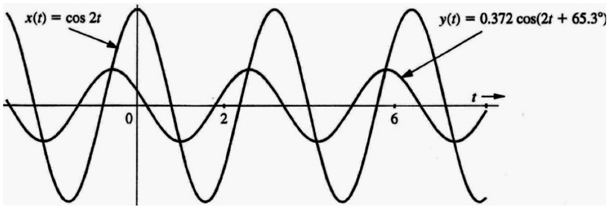
$$H(s)\Big|_{s=j\omega} = H(j\omega) = |H(j\omega)|\, e^{j\,\angle H(j\omega)}$$

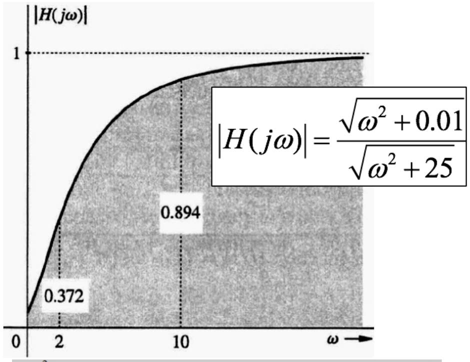
If you go back to the basic definition of Laplace and Fourier transforms, you can immediately see that, assuming causality (i.e. signal is only non-zero for t ≥ 0), then it is clear that you can evaluate the frequency response of a system by substituting s = jω into the transfer function H(s).

It is important to understand the relationship between Fourier and Laplace. Fourier is a special case of Laplace transform for specific value of s.

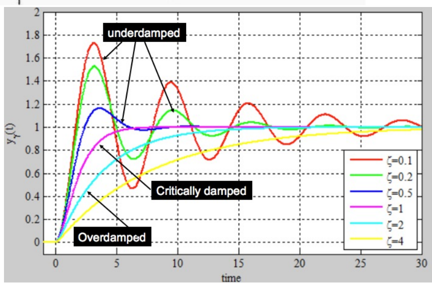
5 – Steady state vs Transient

- When using frequency response H(jω), we assume inputs are everlasting sinusoids at frequency jω. That is, we assume that all transient conditions have died down.





- When using Laplace transform to model system behaviour, we can model both steady state and transient behaviours.



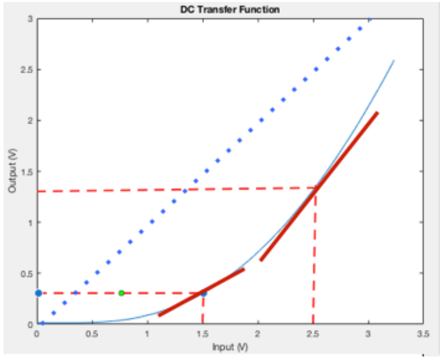
This follows from the last idea. Fourier transform is useful to predict behaviour for everlasting sinusoids. Therefore frequency response H(jω) is useful for studying the steady state behaviour of a system. Steady state means that all transient behaviour has died down.

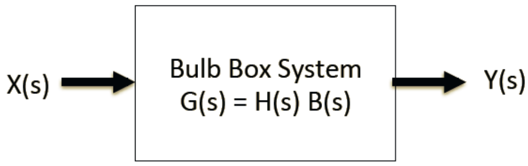
In contrast, Laplace transform and transfer function are useful to model system in both transient condition and steady state condition. That is why we have to learn about Laplace transform.

Studying a system in terms of its transfer function allows use to predict (and understand) its behaviour what input are changing fast. One very important question is to ask: how would the system respond to a unit step input u(t). (This is known as the Step Response.)

6 – Linear approximation in small signals

- A non-linear system such as the Bulb Board can be approximated as linear provided that we only **operate** in small region.
- We called this the "**operating point**" of the system.
- Shown here are two operating points, one at 2.5V input, and another at 1.5V input.
- If we now only use small signal amplitude (sinewaves), then the behaviour of the non-linear system is approximated to be linear.
- Now, we can use transfer functions to model its behaviour as shown here.





$$G(s) = \frac{1000}{0.038s^3 + 1.19s^2 + 43s + 1000}$$

Laplace and Fourier transforms, frequency response and transfer functions, are all applicable ONLY to systems that are linear (i.e. obey principle of superposition) and time-invariant (i.e. its parameters do not change over time).

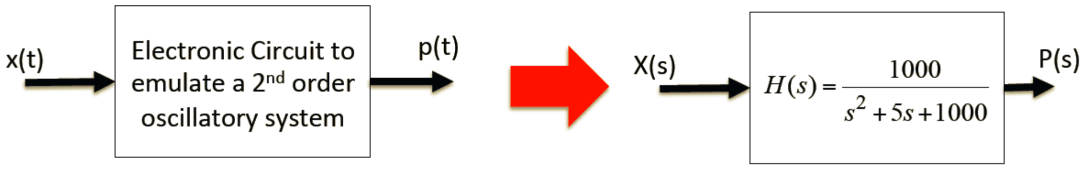
Many real-life systems are non-linear. For example, for the Bulb Box experiment, the output light intensity is a non-linear function of the input voltage (as shown above).

However, in order for us to analyse such as system, we can always operate the system at a certain region, known as the operating point, and use a very small signal. In the experiment, you were studying the system response to a square wave input at Vmin = 1.5v and Vmax = 2.5v. If we assume that the signal is relatively small in amplitude, then we could assume that the system is more-or-less linear. You are now operating along the red lines shown in the graph above.

Under this assumption, they system is regarded as linear. We then use the transfer function shown to model the system. We would however need to adjust the system DC gain, which is lower at 1.5v than at 2.5v.

7 – Damping factor and Natural frequency

- Let us take the transfer function H(s) of the 2nd order system used in Bulb Board is an example:



  - $\omega_0 = \sqrt{a_0} = 31.62$, **resonant frequency** in rad/sec, or 31.62/2π = 5Hz
  - $\zeta = \frac{a_1}{2\sqrt{a_0}} = \frac{5}{2\sqrt{1000}} = 0.079$, the **damping factor** (very small, ideal = 1)
  - $K = \frac{b_0}{a_0} = 1$, DC gain of the system at zero frequency
- Since the damping factor is very small (much smaller than 1), this system is highly oscillatory.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

Why do we bother to model systems in the s-domain at all? The answer is: it provides excellent INSIGHTS to the behaviour of the system in very simply (but fairly accurate) way.

If we take the 2nd order system in the Bulb Box, we have the transfer function H(s) as shown.

We can always re-arrange the denominator equation in terms of the damping factor ζ and the natural frequency ω_0.

ω_0 tells us the frequency at which the system LIKES to oscillate at. ζ tells us the rate at which this oscillation will die down.

The damping factor also characterise the system into various categories. If ζ is >1, then the system is over-damped – it responses to a unit step function slower than it is capable.

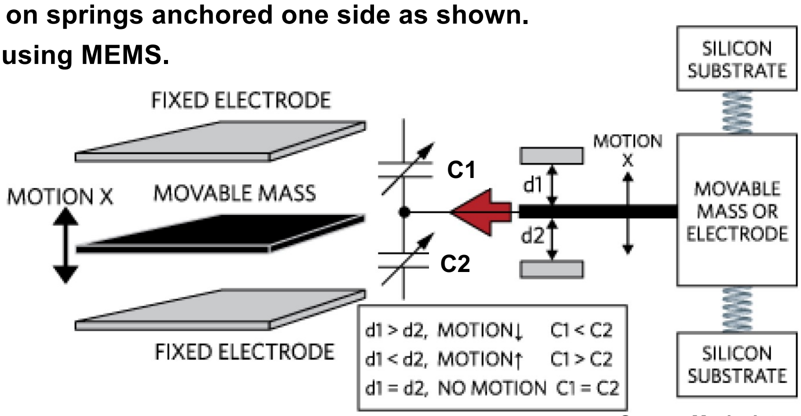
If ζ is < 1, the system is under-damped, and it will oscillate if hit by a transient input.

If ζ is = 1, the system is critically-damped. The system responses fastest without any overshoots or oscillation.

Motion Sensing – Accelerometer

Basic Principle

- **Newton's 2nd Law of motion: F = mass x acceleration.**
- **Sense acceleration is really sensing the force on a mass.**
- **Use capacitive sensing with MEMS.**
- **Acceleration causes mass to move.**
- **Mass pivoted on springs anchored one side as shown.**
- **Implemented using MEMS.**



Source: Maxim Integrated

For the rest of this lecture, the materials are highly integrated with the Lab Session on IMU. You will learn about processing real electrical signals from transducers, and produce estimates of the pitch and roll angles using an electrical system known as a filter.

The IMU is an important component that you will have to use for your self-balancing segway project later. The objective here is for you to understand where the measurements come from, and why you need to process the signal thus produced in order to make it useful.

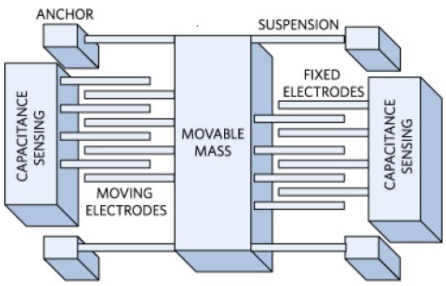
First, let us go through part of what have been covered last year in Electronics 1 during the "Sense" lecture.

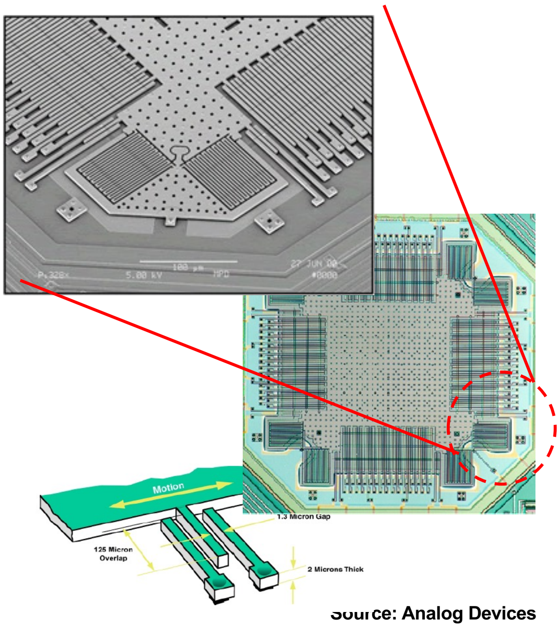
A common sensing approach used in accelerometers is capacitance sensing in which acceleration is related to change in the capacitance of a moving mass. This sensing technique is known for its high accuracy, stability, low power dissipation, and simple structure to build. It is not prone to noise and variation with temperature. Bandwidth for a capacitive accelerometer is only a few hundred Hertz because of their physical geometry (spring) and the air trapped inside the IC that acts as a damper.

Motion Sensing - MEMS accelerometers

Capacitive MEMS accelerometer

- **The displacement of the movable mass (micrometer) is caused by acceleration.**
- **It creates an extremely small change in capacitance for proper detection. Therefore practical sensors use multiple movable and fixed electrodes, all connected in a parallel configuration as shown.**





Source: Analog Devices

The displacement of the movable mass (micrometer) is caused by acceleration, and it creates an extremely small change in capacitance for proper detection. Therefore when implement such accelerometer on MEMS, many parallel capacitors are created as shown in the chip photo here.
The configuration enables a greater change in capacitance, which can both be detected more accurately, and ultimately makes capacitance sensing a more feasible technique.

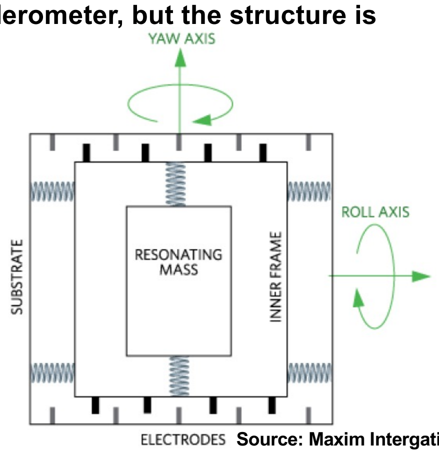
Capacitive MEMS accelerometer

- High precision dual axis accelerometer with signal conditioned voltage outputs, all on a single monolithic IC
- Sensitivity from 20 to 1000 mV/g
- High accuracy
- High temperature stability
- Low power (less than 700 uA typical)
- 5 mm x 5 mm x 2 mm LCC package
- Low cost ($5 ~ $14/pc. in Yr. 2004)

MEMS based accelerometer chips now come with 3-axes, with three separate sensors whose masses move in three orthogonal directions: X, Y and Z.
Makers of accelerometers include: Analog Devices, Bosch, ST Micro, Texas Instruments.

Orientation Sensing - MEMS gyroscopes

- Accelerometers measure linear acceleration (specified in mV/g) along one or several axis.
- A gyroscope measures angular velocity (specified in mV/deg/s).
- Therefore, the accelerometer's output will not respond to change in angular velocity.
- However MEMS gyroscopes are similar to accelerometer, but the structure is different as shown here.
- Here the resonating mass is mounted in an inner frame held by two springs.
- The inner frame is mounted by springs to the substrate with springs in 90 degrees to the inner springs.
- Due to the Coriolis Effect, angular rotations in the roll axis and the yaw axis (see diagram) are now translated to linear accelerations.
- The capacitive fingers are now mounted on the peripherals of the inner frame and the fixed substrate.

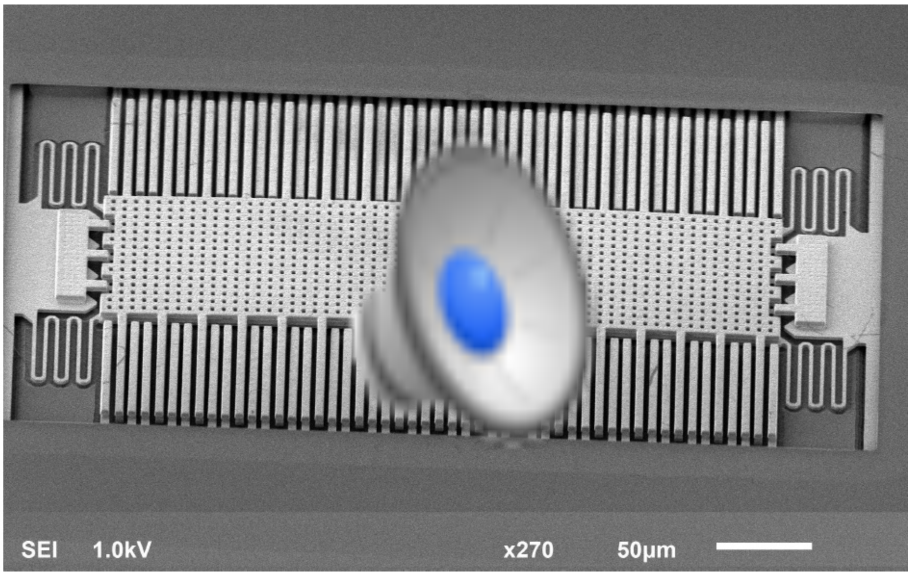


Source: Maxim Intergation

To understand Coriolis effect, here is an excellent video that explains how angular acceleration results in linear acceleration:

https://www.youtube.com/watch?v=mcPs_OdQOYU

A short video on “MEMS Accelerometer”



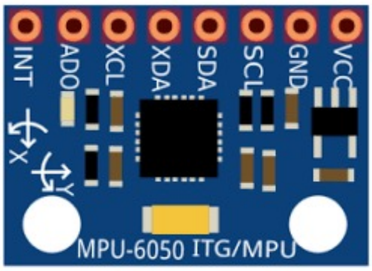
A scanning electron microscope photo of a lateral accelerometer
Piotr Michalik et al, IEEE Senors, Nov 2015

This video can be found on:

https://www.youtube.com/watch?v=i2U49usFo10

Lab 3 – Task 1: Measuring Angel of tilt – the IMU

- The IMU – insertia measurement unit – has built in 3-axis accelerometer and 3-axis gyroscope
- The module uses I2C interface on two pins: SCL and SDA
- Easy to access from Matlab using PyBench:




```
[p, r] = pb.get_accel();        % p, r  = pitch & roll angle in radians
[x, y, z] = pb.get_gyro();      % x, y, z = rate of rotation in 3-axes in rad/sec
```

- **Pitch angle p** – plane pointing up or down
- **Roll angle r** – plane pointing left or right
- Angle can be in unit radian or degree: degrees = radians *180 / π
- Generally use radian for calculations; use degree of display

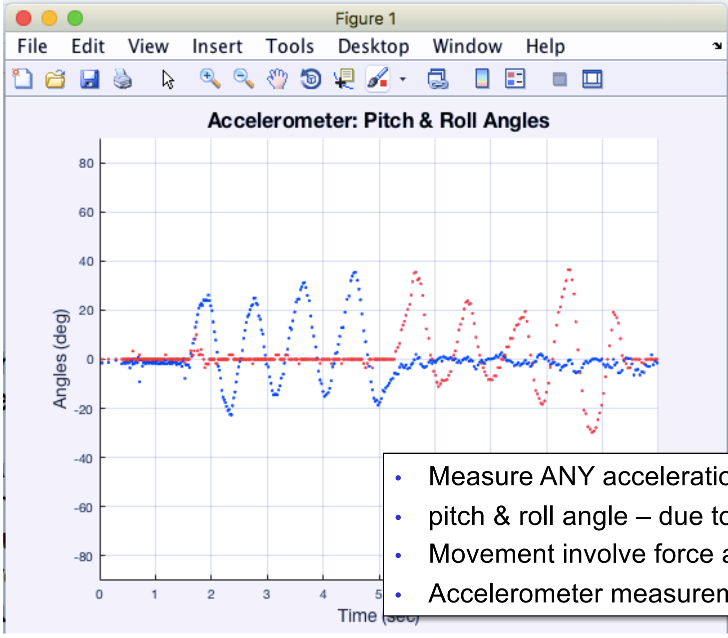
- **x, y** an **z** are the angular velocity in the three axes of rotations

For this course, we will be using a low-cost IMU, the MPU-6050. Datasheet is available on the course webpage. The MEMS device (at the centre of the module) provides accelerometer and gyroscope data as discrete measurements through the I2C interface (see notes from Electronics 1 lecture "Link").

PyBench.m provides two methods to take readings easily: **pb.get_accel()** and **pb.get_gyro()**

The returned values are pitch and roll angles as estimated by the accelerometer, and the angular velocity of rotation from the gyroscope on the x, y and z axes.

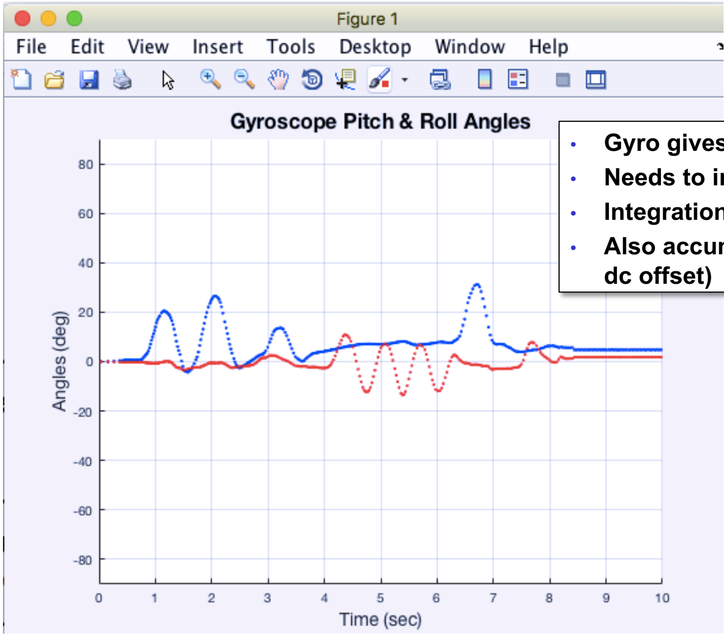
Lab 3 – Task 1a: Accelerometer



- Measure ANY acceleration
- pitch & roll angle – due to gravity g
- Movement involve force acceleration, also measured
- Accelerometer measurement of tilt angle is NOISY

In Lab 3, Task 1, you will be plotting the accelerometer data vs time as pitch and roll angles. You will find that the measurements are “noisy”. This is because accelerometer measures force exerted on a mass due to gravity. However, any other forces (such as those due to vibration or movement) will also manifest themselves as noise on the angle signals.

Lab 3 – Task 1b: Gyroscope

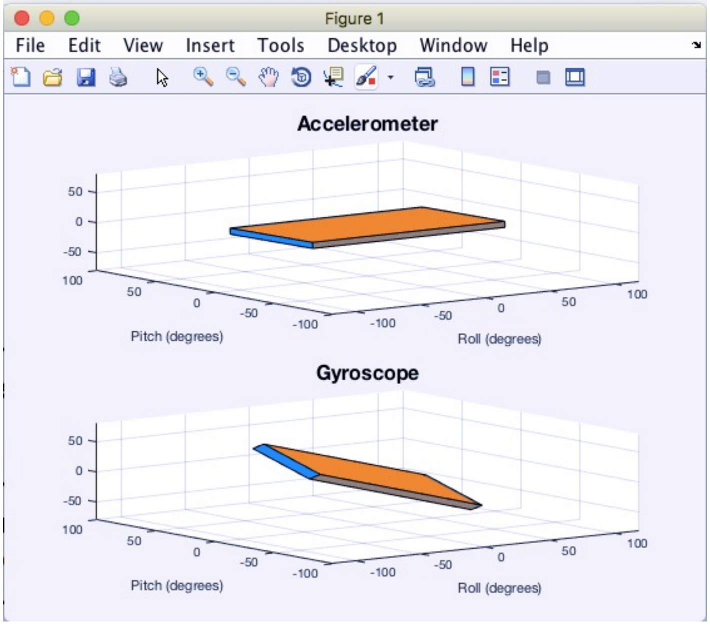


- Gyro gives angular velocity, not angle
- Needs to integrate to get angle
- Integration = accumulation
- Also accumulate errors – causing drift (or dc offset)

When you consider the signals produced by the gyroscope, you need to perform integration of the reading over time in order to obtain the angle reading because the output of the gyroscope is rate of change of angle (or angular velocity). Integration is numerically perform using summation. However, any error in such an operation will produce an error because once summed, the error stays.

The consequence of this way of deriving angles from gryscope readings is that the estimated angle will have an accumulated error over time – something we called a “drift” or an “offset”.

Lab 3 – Task 2: 3D visualization

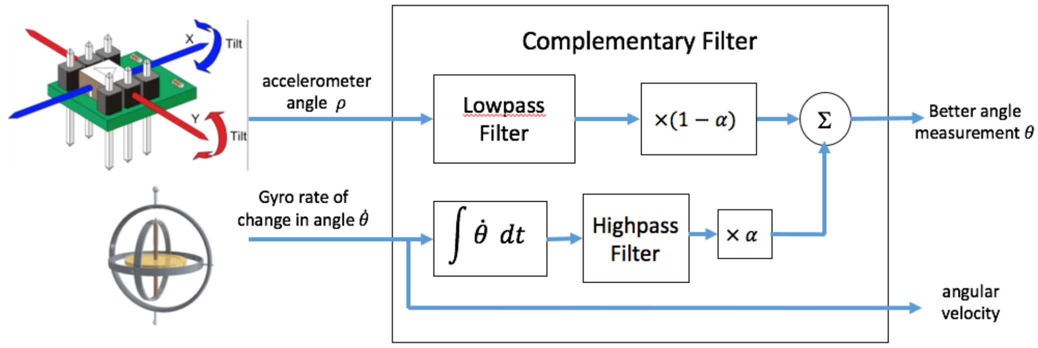


In Lab 3, you will also use a Matlab function I have written to display the IMU board as a 3-D model. This should give you real feel about the measurements from the accelerometer and gyroscope.

The rest of the experiment is to answer the following question:

Given that the accelerometer has noisy measurements, and the gyroscopes data has low noise, but has high drift (increased offset error over time), how can we combine these two readings to give much more reliable pitch and roll angle reading?

Lab 3 – Task 3: Complementary Filter - Concept



$$\text{angle } \theta_{new} = \alpha \times (\ \theta_{old} + \ \dot{\theta}\ dt) + (1-\alpha) \times \rho$$

where
- α = scaling factor chosen by users and is typically between 0.7 and 0.98
- ρ = accelerometer angle
- θ_{new} = new output angle
- θ_{old} = previous output angle
- $\dot{\theta}$ = gyroscope reading of the rate of change in angle
- dt = time interval between gyro readings

I will be going through the complementary filters in details in a later lecture. For now, it is sufficient to see what is the equivalent system we are using to combine these two signals.

Since the accelerometer has lots of noise, we will suppress this with a lowpass filter (pass low frequency components, but suppress high frequency components that contribute to the noise).

Since the gyroscope produces angular velocity that we integrate (sum) to get the angle, and results in high drift (or increasing offset error over time), we will suppress this error using a high pass filter.

Finally we combine these two by adding a bit of both to make up the final angle estimation.

Three Big Ideas

- **Accelerometer** measurement of angle is inherently **noisy** - it cannot distinguish acceleration due to gravity or due to motion.
- **Gyroscope** measurement of angle is inherently "**drifty**" – gyroscope provides angular velocity measurement. Angle measurement is derived through integration. This results in time varying offset called drift.
- Much better angle estimation can be obtained by **filtering and fusion** of the two types of measurements.

Laplace and Fourier transforms, frequency response and transfer functions, are all applicable ONLY to systems that are linear (i.e. obey principle of superposition) and time-invariant (i.e. its parameters do not change over time).

Many real-life systems are non-linear. For example, for the Bulb Box experiment, the output light intensity is a non-linear function of the input voltage (as shown above).

However, in order for us to analyse such as system, we can always operate the system at a certain region, known as the operating point, and use a very small signal. In the experiment, you were studying the system response to a square wave input at Vmin = 1.5v and Vmax = 2.5v. If we assume that the signal is relatively small in amplitude, then we could assume that the system is more-or-less linear. You are now operating along the red lines shown in the graph above.

Under this assumption, they system is regarded as linear. We then use the transfer function shown to model the system. We would however need to adjust the system DC gain, which is lower at 1.5v than at 2.5v.